# CS445 Computational Photography

# Linear Algebra Review

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Original slides by Yuan Shen

- Goals of this session:

- Know systems of linear equations in matrix form
- know basic linear algebra notations (vectors and matrices)
- Know matrix properties (norm, inverse, pseudo-inverse, transpose, rank, eigenvalue, and etc.)
- Know how to setup systems of linear equations and solve them
- Know how to get SVD decomposition of a matrix
- (Eric on Friday) Know how to use Jupyter notebook, numpy

## - Reading Material for Linear algebra

- http://cs229.stanford.edu/summer2019/cs229-linalg.pdf
- YouTube channel (3Blue1Brown) with visualization:
  - https://www.youtube.com/channel/UCYO_jab_esuFRV4b17AJtAw
- CS 357 course website:
  - https://relate.cs.illinois.edu/course/cs357-s19/page/schedule/

- System of Linear Equations in Matrix Form

$$\begin{aligned} 4x_1 &- 5x_2 &= -13 \\ -2x_1 &+ 3x_2 &= 9. \end{aligned}$$

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$$Ax = b$$

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- System of Linear Equations in Matrix Form

$$\begin{aligned} -2x_1 &+ 3x_2 &= 9 \\ 4x_1 &- 5x_2 &= -13 \end{aligned}$$

**Interchanging rows has no effect to solutions**

$$Ax = b$$

$$A = \begin{bmatrix} -2 & 3 \\ 4 & -5 \end{bmatrix}, \quad b = \begin{bmatrix} 9 \\ -13 \end{bmatrix}$$

## - Quiz

- If we know a square matrix $A \in R^{n \times n}$ has n linearly independent eigenvectors, then which of the following is/are true:

  A. The matrix is full ranked

  B. The matrix is invertible

  C. The matrix is diagonalizable

  D. The determinant of A is not equal to 0

  E. The number 0 is not an eigenvalue of A

- Basic Notations:

- The *i-th element of a vector* $x$ *is denoted* $x_i$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

- $A \in R^{m \times n}$, m rows and n columns

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

# Vector

# - Vector Norm

- A norm of a vector ||x|| is informally a measure of the “length” of the vector.

- In particular, l2-norm or Euclidean norm is as follows:

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}.$$

# - Vector-vector inner product

For inner product, $x$ and $y$ should have the same dimension

Given two vectors $x, y \in \mathbb{R}^n$, the quantity $x^T y$, sometimes called the ***inner product*** or ***dot product*** of the vectors, is a real number given by

$$x^T y \in \mathbb{R} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \sum_{i=1}^{n} x_i y_i.$$

# - Vector-vector outer-product

For outer product, $x$ and $y$ do not have to be in the same dimension

Given vectors $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$ (not necessarily of the same size), $xy^T \in \mathbb{R}^{m \times n}$ is called the ***outer product*** of the vectors. It is a matrix whose entries are given by $(xy^T)_{ij} = x_i y_j$, i.e.,

$$xy^T \in \mathbb{R}^{m \times n} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix} = \begin{bmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_n \\ x_2 y_1 & x_2 y_2 & \cdots & x_2 y_n \\ \vdots & \vdots & \ddots & \vdots \\ x_m y_1 & x_m y_2 & \cdots & x_m y_n \end{bmatrix}.$$

# Matrix

- Matrix Multiplication

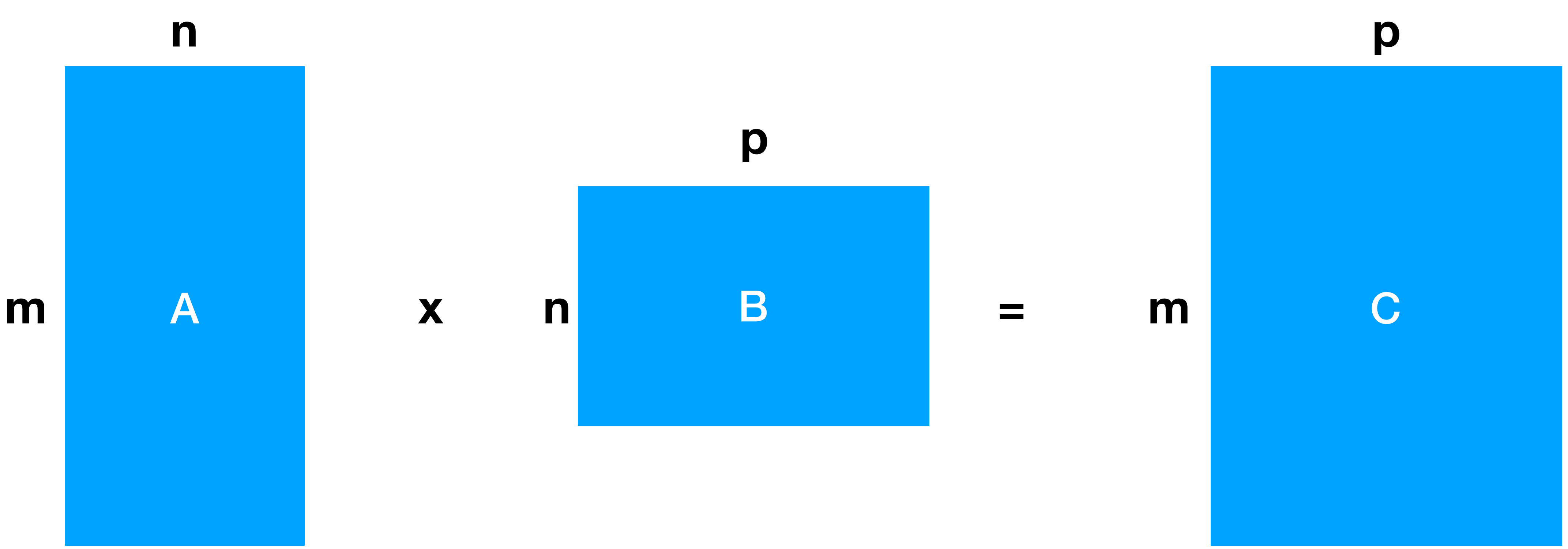
Dimension must match!

The product of two matrices $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$ is the matrix

$$C = AB \in \mathbb{R}^{m \times p},$$

where

$$C_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}.$$



## - Matrix Properties

- Matrix multiplication is associative: $(AB)C = A(BC)$.
- Matrix multiplication is distributive: $A(B + C) = AB + AC$.
- Matrix multiplication is, in general, *not* commutative; that is, it can be the case that $AB \neq BA$. (For example, if $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times q}$, the matrix product $BA$ does not even exist if $m$ and $q$ are not equal!)

- Matrix Transpose

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \quad \rightarrow \quad A^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

The following properties of transposes are easily verified:

- $(A^T)^T = A$
- $(AB)^T = B^T A^T$
- $(A + B)^T = A^T + B^T$

## - Matrix Rank

- Definition: the number of linearly independent columns/rows of A
- For $A \in \mathbb{R}^{m \times n}$, $\text{rank}(A) \leq \min(m, n)$. If $\text{rank}(A) = \min(m, n)$, then $A$ is said to
  ***full rank***.

## - Matrix Inverse

The ***inverse*** of a square matrix $A \in \mathbb{R}^{n \times n}$ is denoted $A^{-1}$, and is the unique matrix such that

$$A^{-1}A = I = AA^{-1}.$$

Note that not all matrices have inverses. Non-square matrices, for example, do not have inverses by definition.

# - Matrix eigenvalue and eigenvector

We usually normalized the norm of eigenvector so that, $\|v\|_2 = 1$

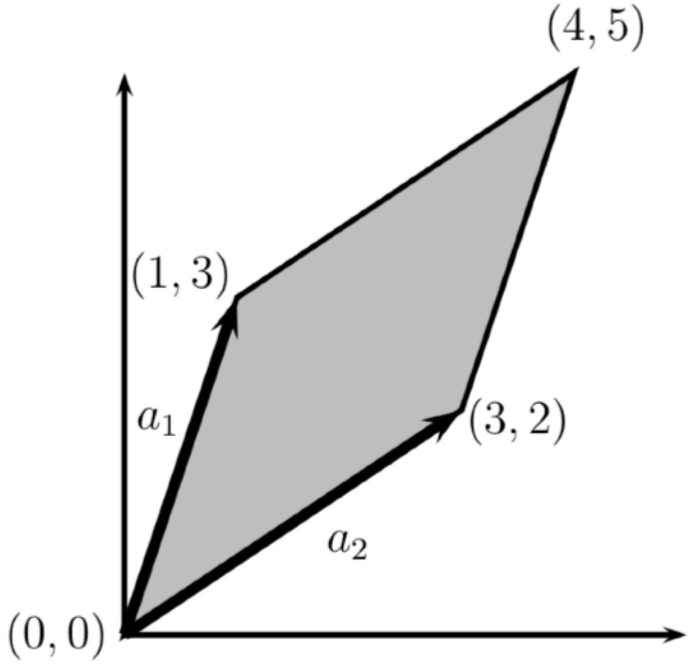
Given a square matrix $A \in \mathbb{R}^{n\times n}$, we say that $\lambda \in \mathbb{C}$ is an ***eigenvalue*** of $A$ and $x \in \mathbb{C}^n$ is the corresponding ***eigenvector***[4] if

$$Ax = \lambda x, \quad x \neq 0.$$

- The determinant of $A$ is equal to the product of its eigenvalues,

$$|A| = \prod_{i=1}^{n} \lambda_i.$$

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}.$$



## - Diagonalizable Matrices

- Definition: For a matrix $A \in R^{n \times n}$, if A has n linearly independent eigenvectors, then A is said to be diagonalizable. In other words,

$$A = UDU^{-1}$$

Also known as Eigendecomposition, spectral decomposition.

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*From CS357 spring 2019*

$$A u_1 = \lambda u_1$$

$$A u_2 = \lambda u_2$$

$$\vdots$$

$$A u_n = \lambda u_n$$

$$\Rightarrow \underbrace{\begin{bmatrix} & A & \end{bmatrix}}_{n\times n} \underbrace{\begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_n \\ | & | & & | \end{bmatrix}}_{n\times n} = \underbrace{\begin{bmatrix} | & | & & | \\ \lambda_1 u_1 & \lambda_2 u_2 & \cdots & \lambda_n u_n \\ | & | & & | \end{bmatrix}}_{n\times n}$$

$$= \begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & \emptyset \\ & \lambda_2 & \\ \emptyset & & \ddots \ \lambda_n \end{bmatrix}$$

$$A U = U D \longrightarrow \boxed{A = U D U^{-1}}$$

Similarity transformation of $A$

$U^{-1}$ exists $\Leftarrow$ Note: $U$ is non-singular since eigenvectors are Linearly indep.

- Quiz

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## - Matrix Singular Value Decomposition

- Definition: factorization a real (or complex) matrix into three matrices.
- $A = U\Sigma V^T$, where U and V are orthogonal matrices, and $\Sigma$ is diagonal matrix.
- Orthogonal (orthonormal) matrix: a real square matrix whose columns and rows are orthogonal unit vectors. The inverse of an orthogonal matrix is its transpose.
- Applications: pseudoinverse, PCA, etc.

- How to calculate the decomposed matrices?

$$A = U\Sigma V^T, A^T = V\Sigma U^T$$

$$A^TA = V\Sigma^T U^T U\Sigma V^T$$

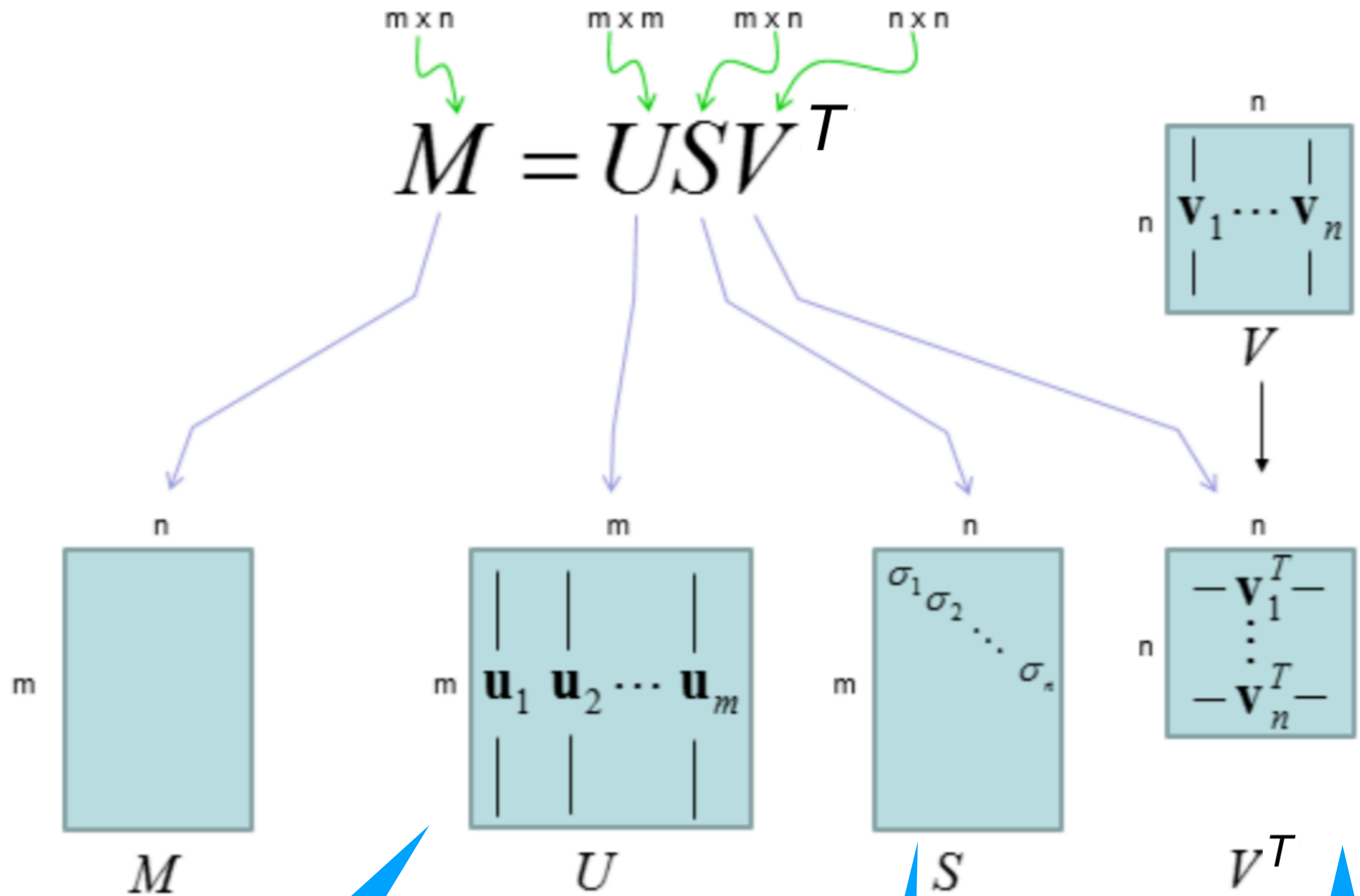
$$A^TA = V\Sigma^T\Sigma V^T$$

Let $D = \Sigma^T\Sigma$

$A^TA = VDV^T$ (diagonalization),

It is in the form of diagonalization! It then indicates that V is the eigenvectors of $A^TA$, and the square root of D gives us the value of $\Sigma$

# - Matrix Singular Value Decomposition (not reduced)




The columns of U are the eigenvectors of $MM^T$

The diagonals of S (singular values) are the square root of eigenvalues of $M^T M$

The columns of V are the eigenvectors of $M^T M$